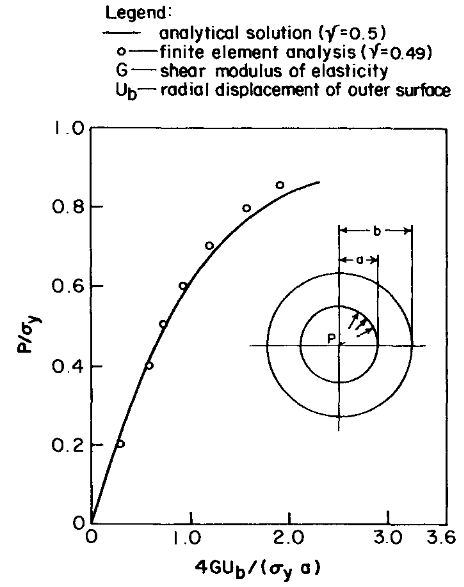

Fig. 4 Variation of radial displacement of a thick wall cylinder.

to plasticity is rather abrupt; high values of η should be used. It is easy to visualize the case $E' = 0$ implies the ideally plastic material and Eq. (1) gives identical results as the model described in Ref. 4.

A computer program was developed by the authors to handle the elastic-plastic stress analysis for the axisymmetric structures by the finite element variational technique. The elastic-plastic stiffness matrix derived by Yamada et al.[5] and the constitutive relation of Eq. (1) served as the basis for such programing. The analytical solution for a thick wall cylinder of linear work hardening material subjected to internal pressure loading given in Ref. 6 was used to check the results obtained by the finite element analysis. The stress-strain relation of the material is assumed to be depicted by the curves shown in Fig. 2 with stress power $n = 50$. The comparison of the two solutions is shown in Figs. 3 and 4. It may be visualized that the agreement of the results is excellent in plastic range whereas discrepancies in the tangential stress distribution occur in the elastic part of the solution. These discrepancies are mainly attributed to the assumption made in the analytical solution that the cylinder material is incompressible. The Poisson's ratio was assumed to be 0.5 throughout the elastic-plastic analysis. This value of Poisson's ratio causes numerical instability in calculating the elastic stiffness matrix in the finite element computations. A lower value of 0.49 had to be used. This number was close to that used in the analytical solution but was still too close to 0.5. A somewhat inaccurate result was thus obtained in the elastic portion of the finite element analysis.

The constitutive equation suggested in Eq. (1) has been proven to be versatile and practical for handling general elastic-plastic stress analysis of solid structures. The tedious iterations for estimating the proper load increments for the structural elements near the elasto-plastic transition region can be avoided by using this simple relation.

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Technical Comments

Comment on "Neighboring Extremals for Optimal Control Problems"

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IN a recent paper, Hymas, Cavin, and Colunga[1] discussed the extension of earlier work on neighboring optimal control for problems without path constraints to problems with a state variable inequality constraint. The purposes of this Comment are to point out a number of errors in their paper, to mention that similar work in this area has been done by Speyer,[2] and to discuss several consequences of the new first-order necessary conditions of Jacobson, Lele, and Speyer[3] which the authors have not considered.

A control constraint on a state constrained arc may be obtained by differentiating the state constraint with respect to time until the control appears explicitly. If q such differentiations are required, the state constraint is said to have order q. Two techniques for handling state constrained problems by means of variational calculus are discussed by Speyer.[2] One involves adjoining the state constraint directly to the Hamiltonian with an undetermined multiplier; the other involves adjoining the control constraint to the Hamiltonian along the constrained arc, and also adjoining a q-dimensional set of entry (or exit) constraints to the performance index. Hymas et al., follow a modified form of the latter approach, using the control constraint to eliminate the control (assumed scalar) immediately, rather than adjoining the control constraint to the Hamiltonian. Hence, their Hamiltonian is defined differently than in Ref. 2. Using Hymas' approach

$$\dot{\lambda}^T = -H_x + H_u Z_u^{-1} Z_x$$

with $H_u \neq 0$, in general, on constrained arcs.

Speyer generalizes the backward sweep method of McReynolds[4] for solving two-point boundary-value problems to the state constrained case, and states jump conditions for the backward sweep matrices (with a few minor mistakes). Similar jump conditions for problems with interior point constraints are stated in greater detail in Ref. 5. Hymas et al. also describe a backward sweep, which includes a number of errors, as described below. All equations cited by number are equations in Ref. 1.

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The lower limits of integration in the second variation, Eqs. (9) and (15), could be more simply chosen to be t_0, t_1^+, and t_2^+, rather than t_0, $t_1^+ + \delta t_1$, and $t_2^+ + \delta t_2$. The resulting expression for the second variation is valid independent of the sign of δt_1, δt_2, and δt_f.

Equations (22) and (24) are missing terms on the right-hand side. By linearizing the corner condition

$$\Psi[x(t_2), u(t_2^-), u(t_2^+), \lambda(t_2^+), t_2] \triangleq L(t_2^-) - L(t_2^+) + \lambda^T(t_2^+)f(t_2^-) - \lambda^T(t_2^+)f(t_2^+) = 0$$

one obtains

$$\Psi_{x_2} dx(t_2) + \Psi_{u_2^-} du(t_2^-) + \Psi_{u_2^+} du(t_2^+) + \Psi_{\lambda_2^+} d\lambda(t_2^+) + \Psi_{t_2}\delta t_2 = 0$$

where

$$\Psi_{x_2} = H_x(t_2^-) - H_x(t_2^+)$$
$$\Psi_{u_2^-} = H_u(t_2^-)$$
$$\Psi_{u_2^+} = H_u(t_2^+) = 0$$
$$\Psi_{\lambda_2^+} = f(t_2^-)^T - f(t_2^+)^T$$
$$\Psi_{t_2} = H_t(t_2^-) - H_t(t_2^+)$$

The differentials appearing above may be replaced by variational quantities as follows, to first order:

$$dx(t_2) = \delta x(t_2^-) + \dot{x}(t_2^-)\delta t_2$$
$$du(t_2^-) = \delta u(t_2^-) + \dot{u}(t_2^-)\delta t_2$$
$$d\lambda(t_2^+) = \delta\lambda(t_2^+) + \dot{\lambda}(t_2^+)\delta t_2$$

Noting that

$$\delta u(t_2^-) = -(Z_u^{-1} Z_x \delta x)_{t=t_2^-}$$

yields

$$0 = \varepsilon_1 \delta x(t_2^-) + \varepsilon_2 \delta t_2 + [\dot{x}(t_2^-)^T - \dot{x}(t_2^+)^T]\,\delta\lambda(t_2^+)$$

where ε_1 and ε_2 are given in Eqs. (11) and (12). The term involving $\delta\lambda(t_2^+)$ is missing in Eq. (22). It can similarly be shown that Eq. (24) is missing the term

$$[\dot{x}(t_1^-)^T - \dot{x}(t_1^+)^T]\,\delta\lambda(t_1^+)$$

on the right-hand side. The same conclusions may be reached by considering the accessory minimization problem, i.e., the minimization of $\delta^2 J$ as given by a corrected version of Eq. (9), subject to linearized constraints (16–18) as well as

$$Z_u \delta u + Z_x \delta x = 0, \quad t_1^+ \leqq t \leqq t_2^-$$
$$\delta x(t_2^-) - \delta x(t_2^+) + [\dot{x}(t_2^-) - \dot{x}(t_2^+)]\delta t_2 = 0$$
$$\delta x(t_1^-) - \delta x(t_1^+) + [\dot{x}(t_1^-) - \dot{x}(t_1^+)]\,\delta t_1 = 0$$

As a result of these missing terms, Eqs. (43, 46, 62, 68, 76, and 85) are incorrect. $d_c(t)$ should be transposed in Eq. (64). With these errors corrected, $d_c(t) \equiv m_c(t)$ and $d(t) \equiv a(t)$. The sweep formulation then becomes symmetric, as it should be (though still inhomogeneous).

It is claimed by the authors that ε_3 and ε_4 must be zero if the control is continuous at the entry point. While this is true in their examples, it is not apparent that this is universally the case. If ε_3 or ε_4 is nonzero, then Eqs. (23) and (24) do not reduce to Eq. (30). The simplified sweep expressions (96–106) are valid only if ε_3 and ε_4 are zero.

Equation (27) is incorrect. Equation (38) is valid only if the control constraint has been adjoined to the Hamiltonian (i.e., if $H \triangleq \lambda^T f + \eta Z$, $t_1^+ \leqq t \leqq t_2^-$, where η is an undetermined multiplier), which is not the approach taken by the authors.

The gain G_1, as given by Eq. (109), is not a control feedback gain on the state, but rather a gain expressing $\delta\lambda$ in terms of δx. The same incorrect gain is given as G_3 in Eq. (111).

Other typographical errors and missing terms, etc., occur in Eqs. (9, 14, 15, 41, 43, 51, 61, 62, 87, 125), and the equations after (92).

In the forbidden region problem, the performance index was chosen to be t_f^2, rather than t_f, since δt_f cannot be determined using Eq. (57) in the latter case. This trick is not necessary, however, since δt_f can be determined from the following expression:

$$\begin{bmatrix}\delta v \\ \delta t_f\end{bmatrix} = \begin{bmatrix} Q & n \\ n^T & \alpha \end{bmatrix}^{-1} \left\{ \begin{bmatrix} dM \\ 0 \end{bmatrix} - \begin{bmatrix} R^T \\ m^T \end{bmatrix} \delta x \right\}$$

for $t_2^+ \leqq t < t_f$. The matrix above is always invertible for $t < t_f$, provided that the trajectory is normal, except at conjugate or focal points relative to the terminal manifold.[6] Equation (57) is a special case of the above result, valid only when $\alpha(t) \neq 0$.

In the abstract, introduction, and conclusion, Hymas et al. refer to necessary conditions for neighboring optimal solutions, but do not identify these in the body of the paper. Presumably, these necessary conditions are that the various sweep matrices exist. While existence of these particular sweep matrices and satisfaction of various convexity conditions is sufficient for the existence of neighboring optimal paths, it is not always necessary.[5,6]

Once the feedback gains expressing $\delta u(t)$ in terms of $\delta x(t)$ and dM have been obtained by means of a backward sweep, the authors describe a means of determining a new optimal control history, corresponding to $\delta x(t_0)$ and dM nonzero, by forward integration. To do this, they first determine $\delta u(t)$ by forward integration of the linearized state Eqs. (18) in conjunction with perturbation feedback control law (107), using feedback gains given by Eqs. (109–111) and (113–118). They choose the wrong sets of feedback gains near t_1 and t_2, however. Feedback gains (109–111) (when corrected) should be used for $t_0 \leqq t \leqq t_1^-$, gains (113–115) should be used for $t_1^+ \leqq t \leqq t_2^-$, and gains (116–118) should be used for $t_2^+ \leqq t \leqq t_f$, even if δt_1 and δt_2 are nonzero. δt_1 and δt_2 influence $\delta u(t)$ only indirectly, through the jump relations

$$\delta x(t_1^+) = \delta x(t_1^-) + [\dot{x}(t_1^-) - \dot{x}(t_1^+)]\,\delta t_1$$
$$\delta x(t_2^+) = \delta x(t_2^-) + [\dot{x}(t_2^-) - \dot{x}(t_2^+)]\,\delta t_2$$

which must be taken into account in the forward integration of Eq. (18).

What is really of interest, however, is not the optimal value of $\delta u(t)$ due to perturbations in the state and terminal constraints, but rather the optimal value of $u(t)$. This optimal value is not necessarily $u_{\text{nominal}}(t) + \delta u(t)$, where $\delta u(t)$ is determined as outlined above, especially near t_1, t_2, and t_f. Hymas et al. use essentially this approach, but recognize potential difficulties near t_1, t_2, and t_f, and propose an extrapolation scheme near these points, if δt_1, δt_2, and δt_f are nonzero. This leaves open the possibility of having to extrapolate infinite gains. Time-to-go guidance[7,8] and min-distance guidance[9] appear to perform better than extrapolation schemes in problems without path constraints. Speyer[2] has discussed the extension of time-to-go guidance to problems with interior point constraints or state variable inequality constraints. Speyer notes also that perturbations away from the constraint boundary, in either direction, cannot be handled deterministically, and proposes a stochastic neighboring optimal guidance technique which is able to do this.

Hymas et al. were unable to think of a simple state constrained problem with discontinuous control at entry and exit points on which to apply their control scheme. There is a good reason for this. Speyer[2] has shown that the control is continuous at entry and exit points of a state constraint of arbitrary order, provided that the Hamiltonian is regular (i.e., $H(x, u, \lambda, t)$ has a unique minimum in u along a given $x(t)$, $\lambda(t)$ trajectory). This is the case in most problems of practical interest. It is furthermore shown in Ref. 3 that u and its first $q-2$ time derivatives are continuous at entry and exit points, for $q \geqq 2$, provided that H is regular.

Additional first-order necessary conditions for a stationary trajectory given in Ref. 3 relate the multiplier η, used to adjoin the control constraint $S^{(q)}(x, u, t) = 0$ to the Hamiltonian, to the multipliers μ. Using Hymas' definition of H, $\eta(t)$ is given by $-H_u(S_u^{(q)})^{-1}$.) The quantities $S^{(i)}$ and $\eta^{(i)}$ below denote the ith time derivatives of S and η. Denoting by μ_i the multiplier used to adjoin the constraint $S^{(i-1)}(t_1) = 0$ to the performance index, the following relationships must hold, if $q \geqq 2$:

$$\mu_i = (-1)^{q-i}\eta^{(q-i)}(t_1^+) \geqq 0, \quad i = 2, \ldots, q$$
$$\eta^{(q-i)}(t_2^-) = 0, \quad i = 2, \ldots, q$$

Neighboring stationary trajectories must therefore satisfy the following equations:

$$\delta\mu_i = (-1)^{q-i}[\delta\eta^{(q-i)}(t_1^+)+\eta^{(q-i+1)}(t_1^+)\,\delta t_1], \quad i = 2,\ldots,q$$

$$\delta\eta^{(q-i)}(t_2^-) = 0, \quad i = 3,\ldots,q$$

$$\delta\eta^{(q-2)}(t_2^-)+\eta^{(q-1)}(t_2^-)\,\delta t_2 = 0$$

For first-order state constraints the above equations are inapplicable.

The above expressions must either be included in the backward sweep or else verified after the $\delta x(t)$ and $\delta u(t)$ histories, etc., have been obtained. $\delta\eta(t)$ may be related to other small quantities without difficulty.[5]

If the control is continuous at t_1 and t_2, δt_2 cannot be directly determined using the backward sweep of Hymas et al. Nor can δt_1, if ε_3 and ε_4 are zero. One approach to determining the new t_1 in real time is to integrate the system equations forward using the new control until $S,\ldots,S^{(q)}$ vanish simultaneously. In fact, if u and its first $q-2$ derivatives are continuous, $S^{(q+1)},\ldots,S^{(2q-2)}$ will also vanish at the new t_1. $S^{(2q-1)}(t_1^-)$ will be the first nonzero derivative (for $q \geqq 2$). t_2, the time of exit from the constraint, could be determined in real time as the time at which $\eta,\ldots,\eta^{(q-2)}$ vanish simultaneously. Hymas et al. describe a technique for determining t_1 and t_2 using Eqs. (126) and (127), which appears to involve monitoring $S,\ldots,S^{(q)}$.

A more direct means of obtaining δt_1 and δt_2 is available, however. Linearizing the relation

$$S^{(2q-2)}(x(t_1^-), u(t_1^-), \lambda(t_1^-), t_1^-) = 0$$

yields

$$S_x^{(2q-2)}(t_1^-)\,\delta x(t_1^-)+S_u^{(2q-2)}(t_1^-)\,\delta u(t_1^-)+ S_\lambda^{(2q-2)}(t_1^-)\,\delta\lambda(t_1^-)+S^{(2q-1)}(t_1^-)\,\delta t_1 = 0$$

Thus, δt_1 may be determined in terms of $\delta x(t_1^-)$, $\delta\lambda(t_1^-)$, and $\delta u(t_1^-)$, provided that $S^{(2q-1)}(t_1^-) \neq 0$. δt_1 may be expressed in terms of $\delta x(t)$ and $\delta\mu$, for $t \leqq t_1^-$, using a backward sweep of the form

$$\delta t_1 = \zeta_1\,\delta x+\zeta_2\,\delta\mu+\zeta_3$$

Differential equations and boundary conditions for ζ_1, ζ_2, and ζ_3 are easily derived. By analogy, δt_2 may be determined in terms of $\delta x(t_2^+)$, $\delta u(t_2^+)$, and $\delta\lambda(t_2^+)$ provided that $S^{(2q-1)}(t_2^+) \neq 0$. A backward sweep expression for δt_2 is also obtainable. If q is large, some of the partial derivatives of $S^{(2q-2)}$ may be a bit messy to evaluate. Boundary arcs in problems with $q > 2$ are seldom (if ever) encountered, however. Situations in which boundary arcs (as opposed to boundary points) cannot exist are discussed in Refs. 3 and 10.

δt_1 and δt_2 could also be determined through the linearized versions of the new necessary conditions of Ref. 3, although these expressions seem less amenable to a backward sweep.

As stated above, modifications are required if $q = 1$. If H is regular, it must be true that

$$\mu_1 = \eta(t_1^+)$$

$$\eta(t_2^-) = 0$$

and

$$\delta\mu_1 = \delta\eta(t_1^+)+\dot{\eta}(t_1^+)\,\delta t_1$$

$$\delta\eta(t_2^-)+\dot{\eta}(t_2^-)\,\delta t_2 = 0$$

The two equations above provide one means of determining δt_1 and δt_2 in the continuous control case. Linearization of the expressions $\dot{S}(t_1^-) = \dot{S}(t_2^+) = 0$ provides another.

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Reply by Authors to L. J. Wood

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THE authors would like to thank L. J. Wood for his comments on our paper. We believe that his comments are, on the whole, constructive and that they add additional perspective to the results that we have obtained. We did overlook the excellent paper by Jacobson, Lele, and Speyer[2] in which new *necessary* conditions for the constrained path-optimum control problem are given. We agree that additional insight can be gained by using these new necessary conditions. However, it does not appear that they are especially useful in the formulation of second-order backward-sweep algorithms.

We believe that the most serious question that Wood has raised is his claim that certain terms are missing from our Eqs. (22) and (24) in Ref. 1. After an exchange of private correspondence with Wood, we believe that we have isolated the point of mutual disagreement. Figure 1 depicts the nominal trajectory and a neighboring trajectory at the constraint boundary intersection. The terms defined in Fig. 1 correspond to their usage in the derivations underlying our paper. The developments in Ref. 1 are then based on the observation that, to second order

$$\delta x(t_1+\delta t_1) = \delta x(t_1)+\delta\dot{x}(t_1^+)\delta t_1 \qquad (1)$$

Since the variational trajectory intersects the constraint at $t_1+\delta t_1$, it is apparent from the figure that it is continuous in its first and second time derivatives at t_1. The use of this observation leads to

$$dx(t_1) = [\dot{x}(t_1^+)+\delta\dot{x}(t_1^+)]\delta t_1+\tfrac{1}{2}\ddot{x}(t_1^+)\delta t_1^2+\delta x(t_1) \qquad (2)$$

and

$$dx(t_1) = [\dot{x}(t_1^-)+\delta\dot{x}(t_1^-)]\delta t_1+\tfrac{1}{2}\ddot{x}(t_1^-)\delta t_1^2+\delta x(t_1) \qquad (3)$$

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